Regatta

Junior League

First Round

1a. Four natural numbers are given. For any triplet of numbers the following is known: their sum is divisible by the one that is middle-sized. Show that some of the numbers are equal.

(None)

1g. A circle with center O is inscribed in the triangle ABC. The point L lies on the extension of the side AB beyond A. The tangent from L intersects the side AC in the point K. Find $\angle KOL$, if $\angle BAC = 50^\circ$.

(None)

1c. Around the table sit several knights, who always tell the truth, and knaves, who always lie. The first person said: "If you don't count me, there are one more knaves than knights here", the second one said: "If you don't count me, there are two more knaves than knights here", and so on until the last one. How many people could be sitting around the table?

(None)

Second Round

2a. All the numbers a, b, c are different. All the lines $y = a^2x + bc$, $y = b^2x + ac$ and $y = c^2x + ab$ go through the same point. Prove that $a + b + c = 0$.

(None)

2g. From the point A_0 black and red rays are drawn with a 7° angle between them. Then a polyline $A_0A_1 \ldots A_{20}$ is drawn (possibly self-intersecting, but with all vertices different), in which all segments have length 1, all vertices with even numbers lie on the black ray and the ones with odd numbers — on the red ray. What is the index of the vertex that is farthest from A_0?

(None)

2c. 5 Aborigines want to cross the river Limpopo in a boat which has place for two people. Initially, each of them has heard a rumour about somebody else, that this person is an Ebola virus carrier. About each person somebody has heard that rumour. If an Aborigine has heard the rumour about a person, he will not sit in the boat with him. The Aborigines do not talk on the banks of the river, but in the boat they exchange all rumours that are known to them. It is possible that they all of them are still able to cross the river?

(None)

Third Round

3a. There are 17 different cups on the table filled with compote. The total amount of dried fruit is 10% of all the compote. Petya and Vasya select and drink one glass each turn (Petya starts) until there is no compote left. Prove that Petya always can guarantee that his dried fruit share differs from 10% no more than the corresponding Vasya's share.

(None)

3g. In the convex quadrilateral $ABCD$, the bisectors of the angles A and C are parallel and intersect the diagonal BD in two distinct points P and Q, so that $BP = DQ$. Prove that the quadrilateral $ABCD$ is a parallelogram.

(None)

3c. Several rooks attack all the white squares of the chessboard 50×50. What is the largest number of black squares that can stay unattacked? (Rook attacks the square which it occupies.)

(None)

Fourth Round

4a. 100 different integer numbers are written on a blackboard. Vasya replaces each number with either its square or its cube. Then Vasya does the process one more time, replacing each number with either its square or its cube (choosing the power randomly each time). What is the smallest number of different numbers that can be written on the blackboard in the end?

(None)

4g. A paper rectangle $ABCD$ ($AB = 3$, $BC = 9$) is folded in such way that vertices A and C coincide. What is the area of the obtained pentagon?

(None)

4c. We call a nine-digit number *good* if it has a digit that can be moved to *another* place so that the new nine-digit number has its digits in a stricly ascending order. How many good numbers are there in total?

(None)

Senior League

First Round

1a. The numbers a and b are chosen in such a way that the graphs of $y = ax - b$ and $y = x^2 + ax + b$ enclose a finite figure of non-zero area. Prove that the origin lies inside this figure.

(None)

1g. All the vertices of a regular polygon lie on the surface of a cube, but its plane does not coincide with any of the cube's faces. What is the maximum possible number of vertices of such a polygon?

(None)

1c. Is it possible to cut a 12345×6789 squared rectangle along square borders into 7890 rectangles with equal diagonals?

(None)

Second Round

2a. The product of all positive divisors of a natural number n (including n) ends in 120 zeros. How many zeros can be at the end of the number n? (List all options and prove that no other exist.)

(None)

2g. The incircle of the triangle ABC touches the side AC at D. The $\angle BDC$ is equal to $60°$. Prove that the inscribed circles of triangle ABD and CBD touch BD at the same point and find the ratio of the radii of these circles.

(None)

2c. 100 Aboriginal people were able to cross the river of Limpopo from the left to the right, in a boat which has place for two people. Initially, each Aborigine has heard a rumour about one or more of the others, that they are Ebola virus carriers. The Aborigine would not sit down together in the boat with somebody who he has heard this rumour about. On the left bank the spreading of the rumours is prohibited, but when the Aborigines reach the right bank, they come out of the boat, and everybody on that bank share all rumours with each other, and only then the boat returns. What is the least possible number of Aborigines that had no rumours at all that they are virus carriers about them?

(None)

Third Round

3a. Baron Munchausen wrote 10 terms on the blackboard and wrote down their sum on a sheet of paper. During one operation he replaced one or several terms on the board with their reciprocals, and again wrote down the sum on the same piece of paper. Is it possible that he wrote down the numbers 1, 2, …, 500 on the piece of paper as a result of 500 such operations?

(None)

3g. Let ABC be an isosceles triangle with $AC = BC$. Let N be a point inside the triangle such that $2\angle ANB = 180^\circ + \angle ACB$. Let D be the intersection of the line BN and the line parallel to AN that passes through C. Let P be the intersection of the angle bisectors of the angles CAN and ABN. Show that the lines DP and AN are perpendicular.

(None)

3c. For any coloring of the squares of the checkered board in black and white, the board is divided into connected coloured regions (in a chess colouring all regions consist of one square). Each step Petya selects one region and repaints it into the opposite colour. The repainted region is glued together with the neighbouring regions of the same color, and the number of regions is reduced. What is the least number of steps that Petya needs to make the painted 13×13 chess board into a monochrome board?

(None)

Fourth Round

4a. Find all the functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all real x:

$$\underbrace{f(f(f\ldots f}_{13}(x)\ldots)) = -x, \qquad \underbrace{f(f(f\ldots f}_{8}(x)\ldots)) = x\,.$$

(None)

4g. On the sides BC, CA and AB of an acute triangle ABC the points A_1, B_1 and C_1 are chosen respectively. The circumscribed circles of the triangles AB_1C_1, BC_1A_1 and CA_1B_1 intersect at P. The points O_1, O_2 and O_3 are the centres of these circles. Prove that $4S(O_1O_2O_3) \geqslant S(ABC)$.

(None)

4c. Prove that the number of ways to colour the edges of an n-sided prism into 4 given colours, in such a way that the edges of each face have all of the colors, does not exceed $8 \cdot 6^{n-1} - 12 \cdot 2^{n-1}$.

(None)